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## MAXIMIN, DISCOUNTING, AND SEPARATING HYPERPLANES

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Is Hartwick's rule a necessary condition for maximin in Solow's [1974] model? Until recently this has been an open question; this is surprising given the prominence of the model.

Cairns and Yang [2000] as well as Withagen, Asheim and Buchholz (this issue) claim that the answer is in the affirmative and claim to provide a formal proof. The latter team argues that the proof by the former is not correct and provides an alternative proof, based on Withagen and Asheim [1998]. Although Cairns and Yang [2000] assert that the methodology of Withagen and Asheim [1998] is "contrived", our proof in this issue is not in dispute. This settles the question: Hartwick's rule is necessary in Solow's model.

Nevertheless there is continued controversy. The main point in Cairns' reply in this issue refers to discounting. In Withagen and Asheim it is assumed (in a very general setting) that an efficient constant utility path is supported by positive utility discount factors

having the property that the integral of the discount factors exists. In the application to Solow's model this assumption needs to be checked. In our contribution in this issue we show that indeed it holds. Here we aim to clarify our approach in some detail and relate it to the concept of separating hyperplanes.

**1. The discount factors correspond to a hyperplane.** Modern microeconomic theory, as originating with Arrow and Debreu (Arrow [1951], Debreu [1951], Arrow and Debreu [1954]) is based on the following result. If technology and individual preferences are both convex, then there exists a hyperplane, containing a feasible allocation, that separates all feasible allocations from those that are preferable. Malinvaud [1953] introduced this mathematical tool to the study of dynamic infinite-horizon discrete-time economies.

Inspired by Koopmans [1951], Arrow and Debreu led a noncalculus revolution in microeconomic theory, entailing that a separating hyperplane may not describe technology and preferences even locally.

If one considers a hyperplane that separates the set of feasible allocations of utility across consumers from those utility allocations that are *socially* preferred, then utility allocations on the hyperplane are equally good in social evaluation, *only* if social welfare is a linear function of the individual utilities. However, such a Samuelson-Bergson welfare function need not be linear in utilities, implying that the economy's primitive objective need not be to maximize a weighted sum of utilities. For example, if the economy's welfare judgments are based solely on the Pareto-criterion, then different utility allocations on the separating hyperplane are incomparable.

In the contributions by Cairns and Yang [2000] and ourselves (this issue), the Samuelson-Bergson welfare function is maximin. In an economy with a finite number of consumers, maximin leads to an efficient allocation with equal utility for all consumers if a utility sacrifice by one consumer can be transformed into an equal utility gain for all other consumers. The extension of this condition to an economy with a continuum of consumers over an infinite time horizon means that there are positive rates of utility transformation at different points in

time,  $\pi(t)$ , such that

$$\int_0^\infty \pi(t) dt < \infty.$$

This is the essential condition in the definition of a *regular maximin path*, due to Burmeister and Hammond [1977] and Dixit et al. [1980].

There are no general results for determining whether regularity is a necessary condition for the solution to a maximin problem in the continuous-time infinite-horizon framework, cf. Mitra [2002]. There are nonregular maximin paths: if the initial capital stock in a one-sector model is at least as large as the Golden Rule size, then no maximin path is regular. Indeed, in our paper in this issue, we show that any maximin path in Solow's [1974] model is regular. We do so under assumptions that are more general than those that previously have been used for this purpose.

The convexity and smoothness of Solow's technology mean that there exists a hyperplane in utility space that is unique and separates any feasible utility path from those that are socially preferable according to the maximin criterion. The discount factors that determine this hyperplane are derived from the solution of the maximin program.<sup>1</sup> Hence, they are *endogenous*. For example, keeping the maximin objective fixed, a different vector of initial stocks or a different technology leads to a different path of discount factors. Any point on the hyperplane that differs from the maximin program is socially strictly less preferred as compared to the maximin program.

Thus, any maximin path in Solow's model is *necessarily* associated with supporting discount factors, implying that the premise of Withagen and Asheim [1998] is *necessary* for any efficient constant consumption path in this model. What is necessary cannot simultaneously be "contrived and inconsistent" as claimed, Cairns and Yang [2000]. On the contrary, the necessity means that any path that does *not* satisfy the premise of Withagen and Asheim [1998] is inconsistent with efficient and constant consumption in Solow's model.

**2. Cairns' misinterpretation of our work.** Instead of acknowledging the mathematical fact that any maximin path in Solow's model is supported by positive utility discount factors—discussed above in the tradition of Arrow, Debreu and Malinvaud—Cairns in his response in

this issue claims that, contrary to us, Cairns and Yang [2000] “avoided” discounting in their analysis of efficient constant consumption paths in Solow’s model. It is, however, a proven fact that any maximin path (and thus, any efficient constant consumption path) in Solow’s model maximizes the sum of discounted utility (or consumption) for appropriately chosen discount factors, cf. Proposition 3 of our paper in this issue.

Cairns argues that “[i]n Withagen and Asheim’s objective the maximin problem is assumed to have been solved and the utility-discount factors ... are assumed to be given, i.e., exogenous,” and writes: “In my view, by maximizing [the sum of discounted utilities] with exogenous discount factors, they posit a utilitarian objective”. The discount factors are, however, *endogenous*, being derived (as we do in Proposition 3 of our paper in this issue) from the maximin path. Subsequent to expression (4) of our paper we make this abundantly clear. And establishing that a maximin path corresponds to maximizing discounted utilities at appropriately chosen discount factors, does not entail “[a] willingness to exchange utility at different times” according to these discount factors (as suggested by Cairns).

Cairns claims that a discussion of a regular maximin path need make no mention of a competitive path. However, as defined by Burmeister and Hammond [1977] and Dixit et al. [1980], any regular maximin path is competitive.<sup>2</sup> He also maintains that Solow’s method of posing the minimum resource use problem “was appropriate in discussing the implications of an energy crisis, but could not be applied in a more general investigation”. However, we reconfirm that the method works in Solow’s model. One must be allowed to use this mathematical method even in times when an energy crisis is not imminent. Finally, he asserts that there is “no ... link [between NNP and the value in the economy] in a maximin problem”. However, Asheim and Buchholz [2002] show that there is indeed such a link in the maximin case.

## ENDNOTES

1. There is a mathematical difficulty. The convex sets in utility space that are separated are infinite dimensional due to the fact that we employ a continuum of consumers over an infinite time horizon. The sets can be seen as subsets of the space of (essentially) bounded Lebesgue measurable functions. The support (the discount factors) of the separating hyperplane is in the dual of this space. The dual

is not the space of Lebesgue integrable functions. So it remains to be proven that the support is indeed integrable, and this is done in our paper in this issue. As one would expect, the proof is trivial.

2. Note that (M.1), (M.2), (M.4), and (M.5) are the competitiveness conditions in Burmeister and Hammond [1977].

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